Exercise 25

Solve the boundary-value problem, if possible.

$$y'' + 16y = 0, \quad y(0) = -3, \quad y(\pi/8) = 2$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$r^2 e^{rx} + 16(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 16 = 0$$

Solve for r.

$$r = \{-4i, 4i\}$$

Two solutions to the ODE are e^{-4ix} and e^{4ix}. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_1 e^{-4ix} + C_2 e^{4ix} \\ &= C_1(\cos 4x - i \sin 4x) + C_2(\cos 4x + i \sin 4x) \\ &= (C_1 + C_2)\cos 4x + (-iC_1 + iC_2)\sin 4x \\ &= C_3 \cos 4x + C_4 \sin 4x. \end{aligned}$$

Apply the boundary conditions to determine C_3 and C_4.

$$y(0) = C_3 = -3$$

$$y\left(\frac{\pi}{8}\right) = C_4 = 2$$

Solving this system of equations yields $C_3 = -3$ and $C_4 = 2$. Therefore, the solution to the boundary value problem is

$$y(x) = -3\cos 4x + 2\sin 4x.$$

Below is a graph of $y(x)$ versus x.